INTRODUCTION TO TOPOLOGICAL DEGREE THEORY

(Teschl, Nirenberg)

Topological method to solve the eq $f(x)=y$ in a bd domain

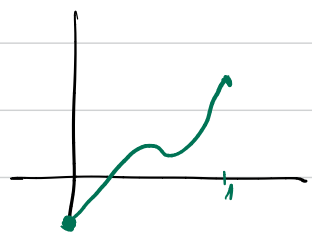
fin dim $\rightsquigarrow$ Brouwer

inf dim $\rightsquigarrow$ Schauder

<u>EXAMPLE</u>: $f:[0,1]\to[-1,1]$ continuous

$f(0)=-1$, $f(1)=1$

$\Rightarrow$ $\forall y\in[-1,1]$, $\exists$ x: $f(x)=y$



Natural question: suppose we can deform $f(x)$ in a continuous way to $f_1(x)$ for which we know we have solutions

$\Rightarrow$ deduce that also $f(x)=y$ has a sol

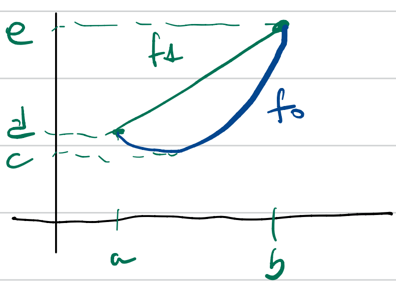
We want a method to "count" solutions stable under deformations

<u>First try</u> (wrong): U open bd set of $\mathbb{R}^n$, $f: U\to\mathbb{R}^n$, cont.

$y\in\mathbb{R}^n$, put

$$N(f,U,y) = \#\{x\in U : f(x)=y\}$$

Is it stable under deformation? No



$$N(f_0,[a,b],y) = \begin{cases} 2 & y\in(c,d] \\ 1 & y\in(d,e]\cup\{c\} \\ 0 & \text{otherwise} \end{cases}$$

$$N(f_1,[a,b],y) = \begin{cases} 1 & y\in[d,e] \\ 0 & \text{otherwise} \end{cases}$$

but $f_0\sim f_1$ via homotopy

So N is not invariant by deformation

look at sol "with signs"

$$\tilde{N} = \#\{x : f(x) = y : f'(x) > 0\} - \#\{x : f(x) = y, f'(x) < 0\}$$

then $\tilde{N}(f_0, [a,b], y) = \begin{cases} 1 - 1 = 0 & y \in (c,d] \\ 1 & y \in (d,e] \\ 0 & \text{otherwise} \end{cases}$ "=" $\tilde{N}(f_1, [a,b], y)$

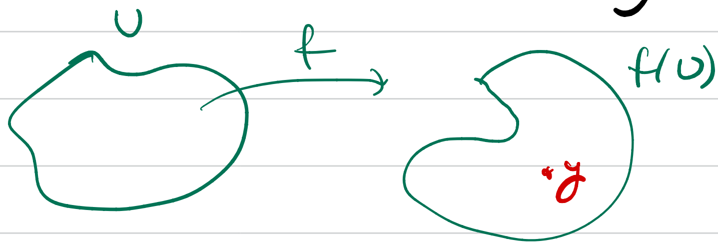
(still problems at $y = c$ and at the bd)

Goal Build a function "counting sol" stable under perturbation and deformations

AXIOMATIC APPROACH : $\mathcal{U} \subseteq \mathbb{R}^n$ open-bounded, $y \in \mathbb{R}^n$

$$D_y(U; \mathbb{R}^n) = \{f : \overline{U} \to \mathbb{R}^n, \text{ continuous}, y \notin f(\partial U)\}$$

($y \notin f(\partial U)$ since we want stability under perturbation)



EXERCISE : $D_y(U; \mathbb{R}^n) \subseteq C^0(U; \mathbb{R}^n)$ is open

Def A degree map is a function

$$\deg(\cdot, U, y) : D_y(U, \mathbb{R}^n) \longrightarrow \mathbb{R}$$
$$f \longmapsto \deg(f, U, y)$$

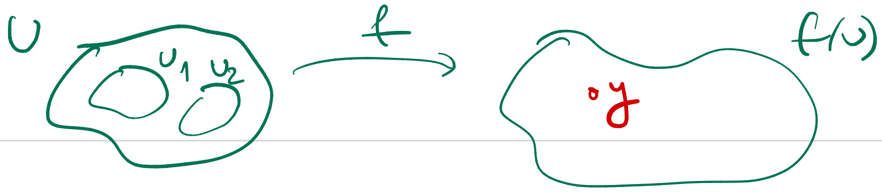
s.t.

(D1) $\deg(f, U, y) = \deg(f - y, U, 0)$ invariance under translation

(D2) $\deg(\mathbb{1}_{\mathbb{R}^n}, U, y) = \begin{cases} 1 & y \in U \\ 0 & y \notin U \end{cases}$ normalization

(D3) $U_1, U_2 \subseteq U$ open, $U_1 \cap U_2 = \emptyset$, $y \notin f(\overline{U} \setminus (U_1 \cup U_2))$

$\Rightarrow \deg(f, U, y) = \deg(f, U_1, y) + \deg(f, U_2, y)$ additivity



(D4) If $h(t,x)$ is <u>admissible homotopy</u>, i.e.

- $h \in C([0,1] \times \overline{U} ; \mathbb{R}^n)$
- $h(t,x) \neq y \quad \forall (t,x) \in [0,1] \times \partial U \qquad \left(\Rightarrow h(t,\cdot) \in D_y(U;\mathbb{R}^n) \ \forall t \right)$

then $\deg(h(t,\cdot), U, y)$ does not depend on t.

If degree exists, it has immediately some interesting properties:

Prop Let deg be given, then

(i) $\deg(f, \emptyset, y) = 0$

(ii) If $y \notin f(\overline{U} \setminus \bigcup_{i=1}^{N} U_i)$, $U_i \cap U_j = \emptyset$ for $i \neq j$

$$\Rightarrow \deg(f, U, y) = \sum_{i=1}^{N} \deg(f, U_i, y)$$

(iii) $f, g \in D_y(U, \mathbb{R}^n)$ and

$\forall x \in \partial U: \quad \mathrm{dist}(f(x), g(x)) < \mathrm{dist}(y, f(\partial U))$

$$\Rightarrow \deg(f, U, y) = \deg(g, U, y)$$

In particular, if $f = g$ on $\partial U \Rightarrow \deg(f, U, y) = \deg(g, U, y)$

(iv) If $\deg(f, U, y) \neq 0 \Rightarrow \exists\, x \in U: \ f(x) = y$
(existence of solutions!)

(v) The map
$$D_y(U, \mathbb{R}^n) \to \mathbb{R}, \quad f \longmapsto \deg(f, U, y)$$
is locally constant

for $f \in D_y(U, \mathbb{R}^n)$ the map
$$\mathbb{R}^n \setminus f(\partial U) \longrightarrow \mathbb{R}, \quad y \longrightarrow \deg(f, U, y)$$
is locally constant

In particular, they are both continuous and constant on the connected components

proof (i) Use (D3) with $U_1 = U$, $U_2 = \emptyset$

(ii) (D3) + induction

(iii) Define: $h(t,\cdot) = (1-t) f + t g$

Check h is admissible: $h(t,x) \neq y \quad \forall (t,x) \in [0,1] \times \partial U$

i.e. $\operatorname{dist}(y, h(t, \partial U)) > 0 \quad \forall t \in [0,1]$.

$$\operatorname{dist}(y, h(t,\partial U)) = \operatorname{dist}(y, f(\partial U)) - \| h(t,\cdot) - f \|_{L^\infty(\partial U)}$$

$$\geq \operatorname{dist}(y, f(\partial U)) - \| f - g \|_{L^\infty(\partial U)} > 0$$

$$\overset{(D4)}{\Longrightarrow} \deg(f, U, y) = \deg(h(0,\cdot), U, y) = \deg(h(1,\cdot), U, y) = \deg(g, U, y)$$

(iv) B.C. $y \notin f(U)$, we also know $y \notin f(\partial U) \Rightarrow y \notin f(\overline{U})$

then (D3) with $U_1 = U_2 = \emptyset$ gives

$$\deg(f, U, y) = \underbrace{\deg(f, U_1, y)}_{=0} + \underbrace{\deg(f, U_2, y)}_{=0 \text{ by (i)}} = 0 \quad ⚡$$

(v) •) $f \in D_y(U; \mathbb{R}^n)$, $g \in C(\overline{U}, \mathbb{R}^n)$ with $\| f - g \|_{L^\infty} \ll \operatorname{dist}(y, f(\partial U))$

$\Rightarrow g \in D_y(U; \mathbb{R}^n)$

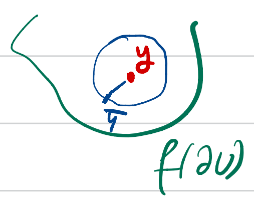
Put $h(t,\cdot) = (1-t) f + t g$ is admissible homotopy

(same proof of (iii))

$\rightsquigarrow \deg(f, U, y) = \deg(g, U, y) \rightsquigarrow$ degree is loc. constant

•) take $\bar{y}$ with $|\bar{y} - y| < \operatorname{dist}(y, f(\partial U))$

put $y(t) := (1-t) y + t \bar{y}$



$\rightsquigarrow \operatorname{dist}(y(t), f(\partial U)) > 0$

$\rightsquigarrow \quad f - y(t) \in D_0(U;\mathbb{R}^n) \quad \forall t \in [0,1]$

$\quad\quad\quad\quad\quad\quad\quad\quad \text{"} \{ g : \ 0 \notin g(\partial U) \}$

$\rightsquigarrow \quad \deg(f, U, y) \overset{(D1)}{=} \deg(f-y, U, 0)$

$\|(f-y(t)) - (f-y)\|$ previous point

$\| y(t) - y\| \leq |\bar{y} - y| << 1$

$$\overset{\text{previous point}}{=} \deg(f - y(t), U, 0) \quad \forall t \in [0,1]$$

$$= \deg(f - \bar{y}, U, 0)$$

$$\overset{(D1)}{=} \deg(f, U, \bar{y})$$

$\square$

FACT: If y is a regular value of $f \in C^1$ then (D1) – (D4) determine deg uniquely

Def ∘) $f : \mathcal{U} \subset \mathbb{R}^n \to \mathbb{R}^n$, $f \in C^1$, $y \in \mathbb{R}^n$ is a <u>regular value</u> of f if $\forall x \in f^{-1}(y)$, $df(x)$ is invertible (if $y \notin f(U)$, then y reg value)

∘) y is critical value if y not regular value

Thm (Sard) $f \in C^1$, the set of critical values has measure 0

Prop U open bd, $y \notin f(\partial U)$ regular value, then $\#\{x \in U : f(x) = y\}$ is finite and if $\{x_1, \dots, x_n\} = f^{-1}(y)$ then $\exists\ U_{x_i}$ neighb of x_i and U_y neigh of y: $f : U_{x_i} \to U_y$ is bijection and $f^{-1}(U_y) = \bigcup_{i=1}^{m} U_{x_i}$

proof $f^{-1}(y)$ closed in $\bar{U}$ $\rightsquigarrow$ it is compact

By the inverse function thm, f loc diffeom around x $\forall x : f(x) = y$

$\rightsquigarrow$ $f^{-1}(y)$ compact set of isolated points

$\rightsquigarrow$ it has finite points. $\square$

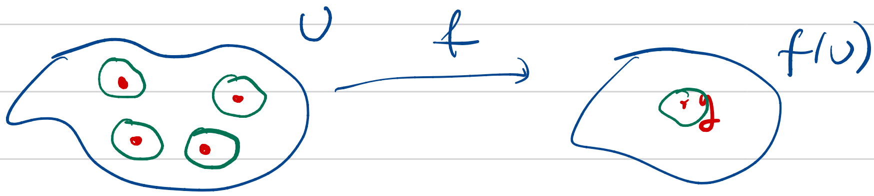
Thm $f \in D_y(U;\mathbb{R}^n)$, $f \in C^1$, y regular value, then any degree map has the form

$$\deg(f,U,y) = \sum_{x \in f^{-1}(y) \cap U} \operatorname{sgn}(\det df(x))$$

(agreement: $\sum_{x \in \emptyset} = 0$)

Rem in $d=1$ this is exactly $\tilde{d}$

proof 1) y reg value, $f^{-1}(y) = \{x_1, \dots, x_n\}$
$\exists\, U_1, \dots, U_n$ open sets st. $y \notin f(\bar{U} \setminus \bigcup_{i=1}^{m} U_i)$



$$\Rightarrow \deg(f,U,y) = \sum_{i=1}^{n} \deg(f, U_i, y)$$

2) We can assume that $U_i = B_{\rho_i}(x_i)$ with $f(x_i) = y$
($df(x_i)$ is invertible, so $f|_{B_{\rho_i}(x_i)}$ loc diffeom, so no other solutions of $f(x) = y$ in U_i)

$$\Rightarrow \deg(f,U,y) = \sum_{i=1}^{n} \deg(f, B_{\rho_i}(x_i), y)$$

3) For $\rho_i \ll 1$, $x \in B_{\rho_i}(x_i)$

$$f(x) = \underbrace{f(x_i)}_{y} + df(x_i)(x - x_i) + o(\|x - x_i\|)$$

put $L_{x_i}(x) = y + df(x_i)(x - x_i)$

We want $\cdot)$ $L_{x_i} \in D_y(B_{\rho_i}(x_i), \mathbb{R}^n)$
$\cdot)$ $\forall x \in \partial B_{\rho_i}(x_i)$: $\operatorname{dist}(f(x), L_{x_i}(x)) < \operatorname{dist}(y, f(\partial B_{\rho_i}(x_i)))$

Prop(ii) $\Longrightarrow$ $\deg(f, B_{\rho_i}(x_i), y) = \deg(L_{x_i}, B_{\rho_i}(x_i), y)$

$\rightarrow$ $\|L_{x_i}(x) - y\| = \|df(x_i)\cdot(x-x_i)\| \geq c\,\|x-x_i\|$

$$\inf_{x\in\partial B_{\rho_i}(x_i)} \|L_{x_i}(x)-y\| \geq c\,\rho_i > 0$$

$\rightarrow$ $\|f(x) - L_{x_i}(x)\| = o(\|x-x_i\|)$

$\forall x\in\partial B_{\rho_i}(x_i)$: $\|f(x) - L_{x_i}(x)\| = o(\rho_i)$

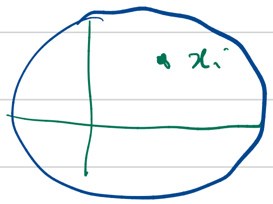
$$\begin{aligned}\forall x\in\partial B_{\rho_i}(x_i):\ \|y - f(x)\| &= \|df(x_i)(x-x_i) + o(\|x-x_i\|)\| \\ &\geq c\,\rho_i - o(\rho_i) \\ &\geq \tfrac{c}{2}\rho_i \quad \text{(provided } \rho_i \text{ suff small)} \\ &\geq \operatorname{dist}(f(x), L_{x_i}(x))\end{aligned}$$

So for $\deg(f, B_{\rho_i}(x_i), y) = \deg(y + df(x_i)(x-x_i), B_{\rho_i}(x_i), y)$

$\overset{(D1)}{=} \deg(df(x_i)(x-x_i), B_{\rho_i}(x_i), 0)$

$\overbrace{df(x_i)(x-x_i)}^{F(x)} = 0 \iff x = x_i$

$\rightsquigarrow F(\overline{B_R(0)} \setminus B_{\rho_i}(x_i)) \not\ni 0$ $\overset{(D3)}{=} \deg(df(x_i)(x-x_i), B_R(0), 0)$, $R \gg 1$



$= \deg(df(x_i)\,x, B_R(0), 0)$

to check this identity: •) $df(x_i)\,x \in D_0(B_R(0), 0)$

•) $\forall x \in \partial B_R(0)$, then

$\operatorname{dist}(df(x_i)(x-x_i), df(x_i)\,x) < \operatorname{dist}(0, \{df(x_i)(x-x_i) : x\in\partial B_R(0)\})$

(exercise, take R suff large)

$\rightsquigarrow$ $$\deg(f, U, y) = \sum_{x\in f^{-1}(y)} \deg(df(x_i)\,x, B_R(0), 0)$$

So we need to understand the degree of a linear map L with $\det L \neq 0$

Lemma Given $L_1, L_2 \in GL(n)$, then they are homotopic inside $GL(n) \iff \mathrm{sgn}(\det L_1) = \mathrm{sgn}(\det L_2)$

proof (Teschl, Lemma 12.3)

Thanks to this lemma, any $L \in GL(n)$ is homotopic to

$$\begin{pmatrix} \mathrm{sgn}(\det L) & & & \\ & 1 & & 0 \\ & & \ddots & \\ & 0 & & 1 \end{pmatrix}$$

Prop If $L \in GL(n)$, then $\deg(L, B_1, 0) = \mathrm{sgn}(\det L)$

proof Since $L \in GL(n)$, then L is homotopic to

$$\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

If $L \sim \mathbb{1} \overset{(D2)}{\Longrightarrow} \deg(\underset{\mathbb{1}_{\mathbb{R}^n}}{\underline{L}}, B_1, 0) = 1 = \mathrm{sgn}(\det L)$

If $L \sim \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$, one argues like this

Idea: construct a function f and sets $U_1 \cup U_2 \subset U$ such that

$$\begin{cases} \deg(f, U, 0) = 0 \\ f|_{U_1} \text{ homotopic to } \begin{pmatrix} -1 & & \\ & 1 & \\ & & \ddots \\ & & & 1 \end{pmatrix} \text{ on } U_1 \\ f|_{U_2} \quad \text{"} \quad \text{"} \quad \mathbb{1} \text{ on } U_2 \\ f^{-1}(0) = \{x_1, x_2\}, \ x_1 \in U_1, \ x_2 \in U_2 \end{cases}$$

then

$$0 = \deg(f, U, 0) = \underbrace{\deg(f|_{U_1}, U_1, 0)}_{-1} + \underbrace{\deg(f|_{U_2}, U_2, 0)}_{1}$$

$\rightsquigarrow \deg\left(\begin{pmatrix} -1 & & \\ & 1 & \\ & & \ddots & \\ & & & 1 \end{pmatrix}, B_1(0), 0 \right) = -1$

(details : Teschl thm 12.4)

(W)

$$\rightsquigarrow \quad \deg(f, U, y) = \sum_{x \in f^{-1}(y)} \operatorname{sgn}\left(\det Df(x)\right)$$

provided y regular value & $f \in C^1$

What about y critical value and $f \in C^0$?

<u>Idea</u> : ·) critical values: let y critical value, pick y_1, y_2 regular values with $|y - y_1|, |y - y_2| << 1$

$\rightsquigarrow \deg(f, U, y_1) = \deg(f, U, y_2)$ (degree loc constant)

define $\quad \deg(f, U, y) = \lim_{k \to \infty} \deg(f, U, y_k)$

with $y_k \to y$, y_k regular value $\forall k$

·) $f \in C^0$: approximate with C^1 functions: given $f \in C^0$, take $f_k \in C^1(U, \mathbb{R}^n) \cap C^0(\bar{U}, \mathbb{R}^n)$ st $f_k \to f$ uniformly in $\bar{U}$. Use that $g \mapsto \deg(g, U, y)$ locally constant to define

$$\deg(f, U, y) = \lim_{k \to \infty} \deg(f_k, U, y)$$

with $f_k \in C^1$, $f_k \to f$ uniformly.

Application of Brouwer degree

Thm (Brouwer's fixed point thm) Let U open set with $\overline{U}$ homeomorphic to $\overline{B_1(0)} \subset \mathbb{R}^n$ and $f : \overline{U} \to \overline{U}$ continuous

$\Rightarrow \exists\, x \in \overline{U} : \quad f(x) = x$

proof Let $\varphi : \overline{U} \to \overline{B_1(0)}$ homeomorphism

then $g = \varphi \circ f \circ \varphi^{-1} : \overline{B_1(0)} \to \overline{B_1(0)}$ continuous

If g fixed point x: $g(x) = x \rightsquigarrow f(\varphi^{-1}(x)) = \varphi^{-1}(x)$

$\rightsquigarrow$ we can assume $U = B_1(0)$

Case 1 $\exists\, x \in \partial B_1(0) : \quad f(x) = x$; nothing to do

Case 2 $f(x) \neq x \quad \forall\, x \in \partial B_1(0)$

strategy: show $\deg(x - f(x), B_1(0), 0) \neq 0 \Rightarrow \exists\, x : f(x) = x$

To compute degree, put

$h(t,x) = x - t f(x)$: $h(0,x) = x = \mathrm{Id}$

$h(1,x) = x - f(x)$

admissible $\Leftrightarrow h(t,x) \neq 0 \quad \forall\, x \in \partial B_1(0) \quad \forall\, t \in [0,1]$

otherwise $x = t f(x)$ for some $x \in \partial B_1(0)$, $t \in [0,1)$

$\rightsquigarrow 1 = |x| = t\,|f(x)| \leq t \quad \rightsquigarrow t = 1$

$\rightsquigarrow x = f(x)$ for some $x \in \partial B_1(0)$ ↯

$$\Rightarrow \deg(x - f(x), B_1(0), 0) = \deg(\underbrace{x}_{\mathrm{Id}}, B_1(0), 0) = 1$$

$\blacksquare$

LERAY-SCHAUDER DEGREE

look for a degree for $F: \mathcal{U} \subseteq X \to X$, X Banach

EXAMPLE: $F: \overset{\ell^2(\mathbb{N})}{B_1(0)} \subseteq \ell^2(\mathbb{N}) \longrightarrow \ell^2(\mathbb{N})$

$$\underline{x} = (x_1, x_2, \dots) \longmapsto \left(\sqrt{1-\|x\|^2}, x_1, x_2, \dots\right)$$

F continuous, $F(B_1(0)) \subseteq \partial B_1(0) = \{x \in \ell^2 : \|x\| = 1\}$

one of consequence of degree theory was Brouwer fixed point

$F: \overline{B_1} \to \overline{B_1}$, assume $\exists$ $x \in \overline{B_1}$: $x = F(x)$

$\leadsto$ $\|x\| = \|F(x)\| = 1$ $\leadsto$ $x = F(x)$

$$\Updownarrow$$

$x_1 = 0$

$x = 0$ ↯ $\Longleftarrow$ $x_2 = x_1 = 0$

$x_j = 0$ $\forall j$

$\leadsto$ Brouwer thm fails in ∞-dim spaces!

Need extra assumption: $F \sim$ compact perturb of $\mathbb{1}$

Def $F: \mathcal{U} \subseteq X \to X$, X Banach, F continuous is said to be <u>compact</u> iff.

$\forall$ $B \subseteq \mathcal{U}$ bd, $\overline{F(B)}$ is compact

<u>Rem</u> $\cdot$) F is nonlinear function. If $F \in \mathcal{L}(X)$ and compact according to def $\leadsto$ old notion of compactness

e) $X = \mathcal{C}^0([0,1])$; $f : \mathbb{R} \to \mathbb{R}$ continuous

$F : X \to X$ $u \mapsto F(u)(t) = \int_0^t f(u(s))\,ds$

F is compact (Ascoli - Arzelà)

Prop let $(F_j)_{j \geq 1}$, $F_j : \mathcal{U} \to X$ compact $\forall j$ and such that $F_j \to F$ in the sup norm to some $F : \mathcal{U} \to X$ continuous.
Then F is compact.

proof let $B \subseteq U$ bounded.

claim $\forall \varepsilon > 0 \ \exists N_\varepsilon \in \mathbb{N}$: $\overline{F(B)} \subseteq \bigcup_{i=1}^{N_\varepsilon} B_\varepsilon(y_i)$

This is true for $\overline{F_j(B)}$ since it is totally bounded.
Now given $\varepsilon > 0$, take j: $\|F - F_j\|_\infty \leq \frac{\varepsilon}{2}$

As $\overline{F_j(B)}$ compact $\Rightarrow \exists$ $\frac{\varepsilon}{2}$-net for $\overline{F_j(B)}$:

$$\overline{F_j(B)} \subseteq \bigcup_{i=1}^{N(\varepsilon)} B_{\frac{\varepsilon}{2}}(y_i)$$

This is ε-net for $\overline{F(B)}$: $\forall y \in F(B)$, $\exists y_a$ $a = 1, \ldots, N(\varepsilon)$ with $\|y - y_a\| < \varepsilon$, indeed

$$\|y - y_a\| \overset{y = F(x),\, x \in B}{=} \|F(x) - y_a\| \leq \|F(x) - F_j(x)\| + \|F_j(x) - y_a\|$$

$$\leq \|F - F_J\|_\infty + \|F_J(x) - y_\alpha\|$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

take the y_α with $F_J(x) \in B_{\frac{\varepsilon}{2}}(y_\alpha)$

$$\rightsquigarrow F(B) \subseteq \bigcup_{i=1}^{N} B_\varepsilon(y_i) \rightsquigarrow \overline{F(B)} \subseteq \bigcup_{i=1}^{N} B_\varepsilon(y_i)$$

□

Cor $F: X \to X$ and $\exists (F_J)_J$ st. $F_J \to F$ in the sup norm and $\forall J$: $\dim(\operatorname{Im} F_J(B)) < \infty$ $\forall B$ bd

$\Rightarrow F$ compact

Rem For linear maps:
- for Hilbert spaces, also converse was true.
- for Banach space, it was false.

Dropping linearity, also converse is valid

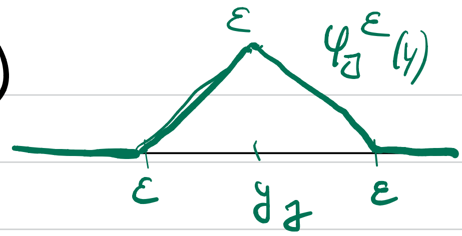
Prop $F: \mathcal{U} \subseteq X \to X$, $\mathcal{U}$ open bounded, and F compact. Then $\forall \varepsilon > 0$ $\exists F_\varepsilon$ continuous such that

$$\|F - F_\varepsilon\| \leq \varepsilon \quad \& \quad \dim(\operatorname{Im} F_\varepsilon) < \infty$$

proof Let $\varepsilon > 0$, $y_1, \dots, y_p$: $\overline{F(U)} \subset \bigcup_{i=1}^{p} B_\varepsilon(y_i)$

and we can choose $(y_i)_{i=1,\dots,p} \in \overline{F(U)}$

Let $\varphi_J^\varepsilon(y) := \max(0, \varepsilon - \|y - y_J\|)$



If $y \in \overline{F(U)}$, $\exists\, J$: $\varphi_J^\varepsilon(y) \neq 0$

$$\rightsquigarrow \quad \psi_J^\varepsilon(y) = \frac{\varphi_J^\varepsilon(y)}{\sum_a \varphi_a^\varepsilon(y)}$$

is well defined for $y \in \overline{F(U)}$ and $\sum_J \psi_J^\varepsilon(y) = 1$

Put
$$F_\varepsilon(x) = \sum_{a=1}^{p} \psi_a^\varepsilon(F(x))\, y_a \qquad (*)$$

•) F_ε continuous function

•) $\operatorname{Im} F_\varepsilon \subseteq \operatorname{span}(y_1, \dots, y_p) \rightsquigarrow F_\varepsilon$ compact

•)
$$\|F(x) - F_\varepsilon(x)\| = \Big\| \underbrace{\sum_{a=1}^{p} \psi_J^\varepsilon(F(x))}_{=1} F(x) - F_\varepsilon(x) \Big\|$$

$$\overset{(*)}{\leq} \sum_{a=1}^{p} \underbrace{\psi_a^\varepsilon(F(x))}_{\neq 0 \text{ only if } \|F(x) - y_a\| < \varepsilon} \overbrace{\|F(x) - y_a\|}^{<\varepsilon} \leq \varepsilon$$

$\sum_{a=1}^{p} \psi_a^\varepsilon(y) = 1$

$$\rightsquigarrow \quad \|F - F_\varepsilon\|_\infty \leq \varepsilon \qquad \blacksquare$$

Lemma $F: \mathcal{U} \subseteq X \to X$, $\mathcal{U}$ open bd and F compact, then $\mathbb{1} + F$ is closed (it maps a closed set into closed sets)

proof $B \subseteq \mathcal{U}$ closed, $(x_n)_n \subseteq B$: $x_n + F(x_n) \to y$

we want to show $y = x + F(x)$.

Since F compact, $\exists\, (x_{n_k})$ s.t. $F(x_{n_k}) \to \bar{y}$

so $x_{n_k} = \underbrace{x_{n_k} + F(x_{n_k})}_{\downarrow \atop y} - \underbrace{F(x_{n_k})}_{\downarrow \atop \bar{y}} \to y - \bar{y}$

$\rightsquigarrow \underbrace{y - \bar{y}}_{\shortparallel \atop x} \in B$ since B is closed

By continuity of F: $x_{n_k} + F(x_{n_k}) \to x + F(x)$
$\searrow y$

$\blacksquare$

Leray - Schauder degree

take $G = \mathbb{1} + F$, F compact

We want to define a degree for $\mathbb{1} + F$ satisfying (D1) - (D4) as in the finite dimensional case

idea: try with finite dimensional approximations

Use prop to approximate F with finite dim range map F_ε acting on finite dim space X_ε : $(\mathbb{1} + F_\varepsilon)|_{X_\varepsilon}$

We shall define the degree as a limit of

$$\deg\left((\mathbb{1} + F_\varepsilon)|_{X_\varepsilon} \right) \rightsquigarrow \text{Brouwer degree of this map}$$

To prove that such a limit is well defined, we need an additional property of Brouwer degree

Idea: $\mathcal{U} \subseteq \mathbb{R}^n$: $f : \overline{\mathcal{U}} \to \mathbb{R}^m$, $m < n$, $y \in \mathbb{R}^m \setminus f(\partial \mathcal{U})$. Put

$$g = \mathbb{1}_{\mathbb{R}^n} + \begin{pmatrix} f \\ 0 \end{pmatrix} \begin{matrix} \} \ m \text{ components} \\ \} \ n-m \text{ components} \end{matrix}$$

Take $y \in \mathbb{R}^m \sim \mathbb{R}^m \times \underbrace{\{0\}}_{n-m \text{ times}} \subseteq \mathbb{R}^n \rightsquigarrow y = \begin{pmatrix} y \\ 0 \end{pmatrix}$

If $x \overset{\in \mathbb{R}^n}{:}$ $g(x) = \begin{pmatrix} y \\ 0 \end{pmatrix} \iff \underset{\substack{\Downarrow \\ x \in \mathbb{R}^m}}{x} + \begin{pmatrix} f(x) \\ 0 \end{pmatrix} = \begin{pmatrix} y \\ 0 \end{pmatrix}$

$\rightsquigarrow \deg(\mathbb{1} + f, \mathcal{U}, y)$ should be the same as $\deg(\mathbb{1} + f|_{\mathcal{U} \cap \mathbb{R}^m}, \mathcal{U} \cap \mathbb{R}^m, y)$

Lemma (Reduction property of Brouwer degree)

Let $f : \overline{\mathcal{U}} \subseteq \mathbb{R}^n \to \mathbb{R}^m$, continuous, $m < n$, $\mathcal{U}$ open, bd, $y \in \mathbb{R}^m \setminus (\mathbb{1} + f)(\partial U)$ then

$$\deg(\mathbb{1} + f, U, y) = \deg((\mathbb{1} + f)|_{\mathcal{U}_m}, \mathcal{U}_m, y)$$

where $\mathcal{U}_m = \mathcal{U} \cap \mathbb{R}^m \equiv \mathcal{U} \cap (\mathbb{R}^m \times \{0\})$

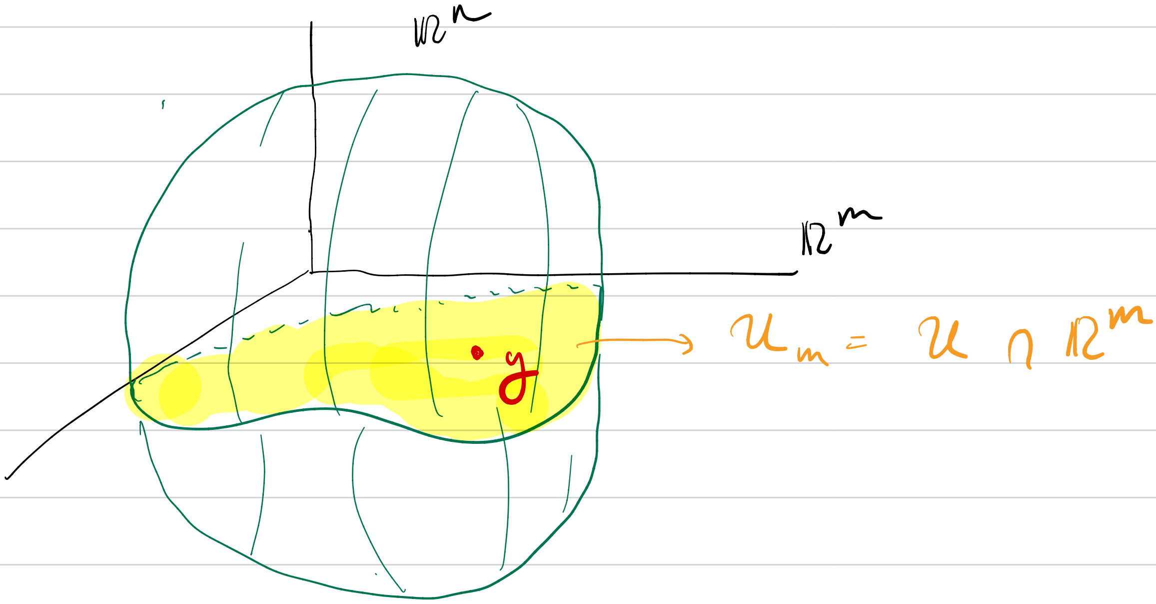
Rem •) $\mathbb{R}^m \simeq \mathbb{R}^m \times \underbrace{\{0\}}_{n-m \text{ components}} \subseteq \mathbb{R}^n$.

$$f: \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

$$x \longmapsto \begin{pmatrix} f(x) \\ 0 \end{pmatrix} \begin{matrix} \}\, m-\text{components} \\ \}\, n-m \text{ components} \end{matrix}$$

•) $\overset{y}{\overset{\shortparallel}{\begin{pmatrix} y \\ 0 \end{pmatrix}}} \in \left(\mathbb{1} + \begin{pmatrix} f \\ 0 \end{pmatrix}\right)(\partial U)$

•)



•) let $x \in \overline{U}$ s.t. $x + f(x) = y \Rightarrow x = \underset{\mathbb{R}^m}{\underset{\cap}{y}} - \underset{\mathbb{R}^m}{\underset{\cap}{f(x)}}$

$\rightsquigarrow x \in \mathbb{R}^m$

$\rightsquigarrow (\mathbb{1} + f)(x) = (\mathbb{1} + f)_{|U_m}(x) = y$

$\rightsquigarrow x \in (\mathbb{1} + f)_{|U_m}^{-1}(y)$

$\Longrightarrow (\mathbb{1} + f)^{-1}(y) = \left((\mathbb{1} + f)_{|U_m}\right)^{-1}(y)$

proof By usual approximation, we can assume $f \in C^1$ and y is regular value. It is enough to prove

$$\operatorname{sign} \det d(\mathbb{1}+f)(x) = \operatorname{sign} \det d\left((\mathbb{1}+f)_{|\mathcal{U}_m}(x)\right)$$

$$\forall\ x \in (\mathbb{1}+f)^{-1}(y) = \left((\mathbb{1}+f)_{|\mathcal{U}_m}\right)^{-1}(y)$$

Recall $f(x) = \begin{pmatrix} f(x) \\ 0 \end{pmatrix}$ $\begin{matrix} m \text{ comp} \\ n-m \text{ comp.} \end{matrix}$

$$\leadsto \det\left(d(\mathbb{1}+f)(x)\right) =$$

$$= \det \begin{array}{c} \left[\begin{array}{c|c} \mathbb{1}_m + \partial_j f_i & \partial_j f_i \\ \hline 0 & \mathbb{1}_{n-m} \end{array}\right] \begin{matrix} \}\, m \\ \}\, n-m \end{matrix} \\ \underbrace{\qquad\qquad}_{m} \quad \underbrace{\qquad}_{n-m} \end{array}$$

develop w.r.t. last $n-m$ rows

$$= \det d(\mathbb{1}_m + f)(x)$$

$\leadsto$ the 2 degrees coincide $\boxtimes$

Define

$$K(\mathcal{U}, X) = \{ F \in C^0(\mathcal{U}, X),\quad F \text{ compact} \}$$

$$F(\mathcal{U}, X) = \{ F \in C^0(\mathcal{U}, X),\quad \operatorname{Im} F \text{ finite dim} \}$$

$$D_y(\mathcal{U}, X) = \{ F \in K(\overline{\mathcal{U}}, X) ; \quad y \notin (\mathbb{1} + F)(\partial U) \}$$

$$\mathcal{F}_y(\mathcal{U}, X) = \{ F \in \mathcal{F}(\overline{\mathcal{U}}, X) : \quad y \notin (\mathbb{1} + F)(\partial U) \}$$

Note that if $F \in D_y(\mathcal{U}, X)$, then

$$\mathrm{dist}\,(y, (\mathbb{1} + F)(\partial U)) > 0$$

(exercise)

Put
$$\rho := \mathrm{dist}\,(y, (\mathbb{1} + F)(\partial U)) > 0$$

then approximate F with $F_1 \in \mathcal{F}(\mathcal{U}, X)$ so that

$$\| F - F_1 \|_\infty < \frac{\rho}{2}$$

$\rightsquigarrow$ $\mathrm{dist}(y, (\mathbb{1} + F_1)(\partial U)) > 0$ $\rightsquigarrow$ $F_1 \in \mathcal{F}_y(\mathcal{U}, X)$

Next, take $X_1 \subset X$ finite dim subspace of X

with $\begin{cases} F_1(\mathcal{U}) \subset X_1 \\ y \in X_1 \end{cases}$

and set $\mathcal{U}_1 := \mathcal{U} \cap X_1$, then we

have also $F_1 \in \mathcal{F}_y(\mathcal{U}_1, X_1)$

We put

$$\left[\ \deg(\mathbb{1}+F,\,\mathcal{U},\,y) := \deg(\mathbb{1}+F_1,\,\mathcal{U}_1,\,y)\ \right]$$

LERAY - SCHAUDER DEGREE

Prop The Leray - Schauder degree is well posed.

proof Pick $F_2 \in \mathcal{F}(\overline{\mathcal{U}}, X)$: $\|F_2 - F\|_\infty < \rho/2$

X_2 as before

and define
$$X_0 := X_1 + X_2$$
$$U_0 = \mathcal{U} \cap X_0$$

By reduction thm ($y \in X_1$, $y \in X_2$)

$$\deg(\mathbb{1} + F_1, \mathcal{U}_0, y) = \deg(\mathbb{1}+F_1, \mathcal{U}_1, y)$$
$$\deg(\mathbb{1} + F_2, \mathcal{U}_0, y) = \deg(\mathbb{1}+F_2, \mathcal{U}_2, y)$$

Put
$$H(t) = \mathbb{1} + (1-t)F_1 + tF_2$$

It is admissible homotopy since

$$\|H(t) - (\mathbb{1}+F)\|_\infty \leq \|F_1 - F\|_\infty + \|F_2 - F\|_\infty < \rho$$

$$\rightsquigarrow \quad \deg(\mathbb{1}+F_1, \mathcal{U}_1, y) = \deg(\mathbb{1}+F_1, \mathcal{U}_0, y)$$

homotopy invariance of Brouwer degree
$$= \deg(\mathbb{1}+F_2, \mathcal{U}_0, y)$$
$$= \deg(\mathbb{1}+F_2, \mathcal{U}_2, y)$$

Thm $\mathcal{U}$ open bd, $\mathcal{U} \subseteq X$, $F \in D_y(\mathcal{U}, X)$, $y \in X$
then Leray – Schauder degree fulfills (D1) – (D4)

proof (exercise :)

Cor The additional properties of Brouwer degree deriving from (D1) – (D4) holds true for Leray - Schauder degree.

Application : Schauder fixed point thm

Thm let D a closed convex bd subset of X Banach and
$$F : D \to D \quad \text{compact}$$
then F has a fixed point (i.e. $\exists\, x \in D$: $F(x) = x$)

proof Assume that $0 \in D$ (otherwise translate the set)

Case 1 If $\exists\, x \in \partial D$ with $F(x) = x$ ✓

Case 2 $\forall x \in \partial D$: $F(x) \neq x$

So we can define $\deg(\mathbb{1} - F, D, 0)$ and show to

Put $\quad h(t,x) = x - tF(x), \quad t \in [0,1], \ x \in D$

$\forall t \in [0,1]$, $h(t,\cdot) = \mathbb{1} +$ compact

Let us show h is admissible: we claim that

$$h(t,x) \neq 0 \quad \forall \ (t,x) \in [0,1] \times \partial D$$

otherwise: $\exists \ (\bar t, \bar x) \in [0,1] \times \partial D$ with $h(\bar t, \bar x) = 0$, i.e.

$$\bar x = \bar t \, F(\bar x)$$

- $F(x) \neq x \quad \forall x \in \partial D \Rightarrow \bar t < 1$
- $F(D) \subseteq D \Rightarrow F(\bar x) \in D$
- D convex $\Rightarrow \bar t \, F(\bar x) \in D$
- $\bar t < 1 \Rightarrow \bar t \, F(\bar x) \in \mathring{D}$ $\lightning$

$\partial D \ni \bar x = \bar t \, F(\bar x)$

$\leadsto h(t,\cdot)$ admissible

$$\Rightarrow \deg(h(0,\cdot), D, 0) = \deg(h(1,\cdot), D, 0)$$

$$\deg(\mathbb{1}, D, 0) \overset{(D2)}{=} 1 \ \ (0 \in D), \qquad \deg(h(1,\cdot), D, 0) = \deg(\mathbb{1} - F, D, 0)$$

$\Rightarrow \deg(\mathbb{1} - F, D, 0) \neq 0 \Rightarrow \exists$ sol of $(\mathbb{1} - T)(x) = 0$

$\blacksquare$

Applications!

Peano Theorem

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases} \quad , x \in \mathbb{R} \;, y \in \mathbb{R}^n$$

open bounded

$f : \mathcal{U} \to \mathbb{R}^n$ of class C^0, and $(x_0, y_0) \in \mathcal{U}$

$\Rightarrow \exists\, \delta > 0$ and a sol $y(x) : [x_0 - \delta, x_0 + \delta] \to \mathbb{R}^n$ to the Cauchy problem

proof idea:

$$(Ty)(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt$$

T is compact from $C^0 \to C^0$

exercise: which is the set D?
Find it and apply Schauder!